

Localized perturbations of integrable quantum billiards

Saar Rahav¹ and Shmuel Fishman^{1,2,3}

¹*Department of Physics, Technion, Haifa 32000, Israel*

²*Department of Physics, University of Maryland, College Park, Maryland 20742*

³*Institute for Theoretical Physics, University of California at Santa Barbara, Santa Barbara, California 93106*

(Received 28 November 2001; revised manuscript received 5 February 2002; published 26 June 2002)

The statistics of energy levels of a rectangular billiard that is perturbed by a strong localized potential are studied analytically and numerically, when this perturbation is at the center or at a typical position. Different results are found for these two types of position. If the scatterer is at the center, the symmetry leads to additional contributions, some of which are related to the angular dependence of the potential. The limit of the δ -like scatterer is obtained explicitly. The form factor, which is the Fourier transform of the energy-energy correlation function, is calculated analytically, in the framework of the semiclassical geometrical theory of diffraction, and numerically. Contributions of classical orbits that are nondiagonal are calculated and are found to be essential.

DOI: 10.1103/PhysRevE.65.067204

PACS number(s): 05.45.Mt, 03.65.Nk, 03.65.Sq

The distribution of energy levels exhibits a high degree of universality and is a central subject in the field of quantum chaos [1,2]. For systems that are chaotic in the classical limit the statistics are those of random matrix theory [3], while for typical integrable systems the level distribution satisfies Poissonian statistics [4]. In the semiclassical regime this universal behavior holds for a wide range in energy. There are also regimes of energy where spectral correlations related to periodic orbits are important [5,6]. In intermediate situations such a high degree of universality is not found. For mixed systems, where in some parts of phase space the motion is chaotic and in other parts it is regular, the statistics exhibit some general features [7,8]. Another type of intermediate behavior may be found for integrable systems perturbed by singularities of spatial extension that are much smaller than the wavelength of the quantum particle. Examples of relevant systems are billiards with flux lines, sharp corners, and δ -like interactions [9–11]. Here we report results obtained for a rectangular billiard perturbed by a δ -like impurity [12], known as the Seba billiard [9]. Some of these results can alternatively be concluded from a recent general formulation by Bogomolny and Giraud [13].

The interest in billiards of various types is primarily theoretical since it is relatively easy to analyze them analytically and numerically. Billiards have been studied also experimentally for electrons [14], microwaves [15], and laser cooled atoms [16]. We hope that in the future perturbations of the type discussed in the present work will also be introduced experimentally.

Trace formulas that express the quantum density of states in the semiclassical limit as sums over classical periodic orbits were derived for chaotic [17,18] and integrable [19] systems. For perturbations smaller than the wavelength, standard semiclassical theory used in the derivation of these formulas fails and diffraction effects have to be taken into account. This can be done in the framework of the geometrical theory of diffraction [20]. In this approximation, which is valid far from the perturbation, the Green's function for the system (without the boundary) is given by $G(k; \mathbf{r}, \mathbf{r}') \approx G_0(k; \mathbf{r}, \mathbf{r}') + G_0(k; \mathbf{r}, \mathbf{r}_0)D(\theta, \theta')G_0(k; \mathbf{r}_0, \mathbf{r}')$, where θ

and θ' denote the directions of $\mathbf{r} - \mathbf{r}_0$ and $\mathbf{r}_0 - \mathbf{r}'$, respectively, and G_0 is the free Green's function. The diffraction constant D describes the scattering from the perturbation. For the rectangular billiard with a δ -like perturbation, which is the subject of the present work, *only* diffraction effects are responsible for the deviations from the behavior of integrable systems. Therefore this is an ideal system for the exploration of such effects. Moreover, for this problem the analytical and numerical calculations are relatively easy. The statistics depend on the location of the scatterer and on the boundary conditions [11,21]. This is in contrast to chaotic systems where the spectral statistics are not affected by such scatterers [22].

The diagonal approximation [5], where only contributions from orbits with equal actions are considered, is extensively used in the field of quantum chaos. It is not applicable for systems with localized perturbations. A method to take into account dominant nondiagonal contributions in integrable systems was developed by Bogomolny [23] and will be used here.

In this work a rectangular billiard with sides a_x and a_y , such that the aspect ratio is irrational, perturbed by a localized scatterer is studied. The scatterer is represented by a potential of typical size a such that

$$U(\mathbf{r}) = \frac{1}{a^2} f\left(\frac{\mathbf{r}}{a}\right) \quad (1)$$

where $f(\mathbf{y})$ is small where \mathbf{y} is large.

The diffraction constant is the on-shell matrix element of the \mathbf{T} matrix $D(\theta', \theta) = \langle \mathbf{k} | \mathbf{T}(E) | \mathbf{q} \rangle$ where \mathbf{k} is the outgoing momentum (in the direction θ') and \mathbf{q} is the incoming momentum (in the direction θ). The energies of the incoming and outgoing waves are equal, that is, $k = q = \sqrt{E}$ (in the units $\hbar = 1$ and $m = \frac{1}{2}$ used in this Brief Report). The Born series cannot be used to compute $D(\theta', \theta)$ when $ka \ll 1$ since the free Green's function diverges as $\ln ka$ at short distances. A method that is regular when $ka \ll 1$ was introduced by Noyce [24]. In this method the scattering in the forward di-

rection is resummed [24]. It leads to a diffraction constant that is a ratio of series. The series in the numerator and in the denominator are expanded in the number of scattering events (just like the Born series). Every term in these series is then expanded for $ka \ll 1$ (up to terms of order $k^2 a^2$) and both series are summed (with respect to the number of scattering events) to give the angle dependent diffraction constant [12]

$$D(\theta', \theta) \approx C \left[1 + i \frac{ka}{2} \{ (e^{i\theta} - e^{i\theta'}) M_1 + \text{c.c.} \} - k^2 a^2 \left(M_0 + \{ (e^{2i\theta} + e^{2i\theta'}) M_2 + \text{c.c.} \} - \sum_{c,d=-1,+1} M_{cd} e^{i(c\theta' + d\theta)} \right) \right], \quad (2)$$

where

$$C \equiv \left\{ \frac{V(1)}{V_0} + \frac{i}{4} - \frac{1}{2\pi} \left[\gamma + \ln \left(\frac{ka}{2} \right) \right] + k^2 a^2 \frac{Q(1)}{V_0} \right\}^{-1}, \quad (3)$$

c.c. denotes the complex conjugate, γ is Euler's constant, and $V_0 = \int d^2 y f(\mathbf{y})$. Also $V(1)$, $Q(1)$, M_0 , M_1 , M_2 , and M_{cd} are constants, independent of θ, θ' , and M_{cd} depend logarithmically on ka . These constants, given by series of integrals, involving the potential (1), were calculated in [12].

In the limit $a \rightarrow 0$ and k fixed, a finite diffraction constant is obtained if the potential is such that $V(1)/V_0 \sim (1/2\pi)[\ln(a/l) + B]$, where l and B are constants, leading to

$$D \approx 2\pi \left[\frac{i\pi}{2} - \ln \left(\frac{kl}{2} \right) - \gamma + B \right]^{-1}. \quad (4)$$

It depends on the combination $B - \ln l$ of the two parameters l and B . Therefore these are somewhat arbitrary in the limit $a \rightarrow 0$.

First we assume that a is sufficiently small so that Eq. (2) can be approximated by Eq. (4), and because of its slow variation with k it can be replaced by the constant D . The oscillatory part of the density of states, in the semiclassical limit, is a sum over contributions of periodic and diffracting orbits. Diffracting orbits are orbits that start and return to the scatterer. For the rectangular billiard with a localized (angle independent) scatterer at its center the density of states is [12]

$$d_{osc}(E) = \sum_p A_p^{(0)} e^{ikl_p} + \sum_{j_1} A_{j_1}^{(1)} e^{ikl_{j_1}} + \sum_{j_1, j_2} A_{j_1, j_2}^{(2)} e^{ik(l_{j_1} + l_{j_2})} + \sum_{j_1, j_2, j_3} A_{j_1, j_2, j_3}^{(3)} e^{ik(l_{j_1} + l_{j_2} + l_{j_3})} + \dots + \text{c.c.}, \quad (5)$$

where

$$A_p^{(0)} = (2\mathcal{A}/\pi\sqrt{8\pi kl_p}) e^{-i\pi/4},$$

$$A_{j_1}^{(1)} = [(-1)^{x_1} \sqrt{l_{j_1}}/\pi k \sqrt{8\pi k}] D e^{-i(3/4)\pi},$$

$$A_{j_1, j_2}^{(2)} = (-1)^{x_2} [(l_{j_1} + l_{j_2})/4\pi^2 k^2 \sqrt{l_{j_1} l_{j_2}}] D^2 e^{-i(3/2)\pi},$$

and

$$A_{j_1, j_2, j_3}^{(3)} = (-1)^{x_3} [16D^3/3\pi k (8\pi k)^{3/2}] \times (l_{j_1} + l_{j_2} + l_{j_3})/\sqrt{l_{j_1} l_{j_2} l_{j_3}} e^{-i\pi/4},$$

where $x_1 = N_{j_1} + M_{j_1}$ and $x_i = x_{i-1} + N_{j_i} + M_{j_i}$. The area of the billiard is \mathcal{A} , the length of a periodic orbit is l_p , and l_j is the length of a diffracting segment j with N_j and M_j reflections from the boundary. The density of states (5), which is expanded to the third order in D , is used to compute the correlation function

$$R_2(\eta) = \left\langle d_{osc} \left(E - \frac{\eta\Delta}{2} \right) d_{osc} \left(E + \frac{\eta\Delta}{2} \right) \right\rangle \Delta^2 \quad (6)$$

and its Fourier transform, the form factor

$$K(\tau) = \int_{-\infty}^{\infty} d\eta R_2(\eta) e^{2\pi i \eta \tau}, \quad (7)$$

where Δ denotes the mean level spacing. The angular brackets denote averaging over an energy scale much larger than Δ but much smaller than E . If only the contributions from periodic orbits to the density of states are taken into account the form factor is given by

$$\frac{K(\tau)}{2\pi\Delta} = \left\langle \sum_{pp'} A_p A_{p'}^* e^{i(S_p - S_{p'})} \delta \left(\frac{2\pi}{\Delta} \tau - \frac{t_p + t_{p'}}{2} \right) \right\rangle \quad (8)$$

where S_p is the action of the orbit, t_p is its period, and $\tau > 0$ is assumed. The diagonal approximation can be used to compute the contributions from periodic and once diffracting orbits. When there are more than three segments of orbits in the exponent of Eq. (8), nondiagonal contributions are of importance, since then one finds saddle manifolds consisting of different combinations of orbits with almost identical total length so that their phase is almost stationary [23]. An example of such a saddle manifold is given by a periodic orbit of length $l_p = 2\sqrt{N_p^2 a_x^2 + M_p^2 a_y^2}$, and the pairs of diffracting segments of lengths $l_{j_i} = \sqrt{N_{j_i}^2 a_x^2 + M_{j_i}^2 a_y^2}$ that satisfy $N_{j_1} + N_{j_2} = 2N_p$, $M_{j_1} + M_{j_2} = 2M_p$, and $N_{j_i}/M_{j_i} \approx N_p/M_p$. The length difference $l_p - l_{j_1} - l_{j_2}$ is small, of the order of $1/k$. Since in billiards the action is $S_j = kl_j$ the action difference is of order unity and these contributions are in phase. Other nondiagonal contributions of this type can contribute significantly as well. The resulting form factor for a scatterer at the center, up to order τ^3 , is found to be [12]

$$K(\tau) = 1 - \frac{|D|^2}{4} \tau + \frac{1}{8} |D|^4 \tau^2 + \left(\frac{1}{2} |D|^4 - \frac{1}{24} |D|^6 \right) \tau^3. \quad (9)$$

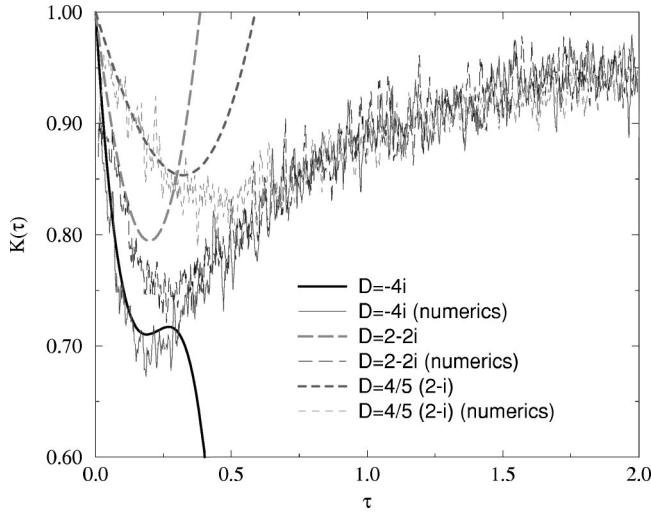


FIG. 1. The form factor of a scatterer at a typical position, for some diffraction constants, thin lines, compared to the analytical result (11), heavy lines.

To obtain Eq. (9) we used the optical theorem, which for angle independent scattering is

$$\text{Im} D = -\frac{1}{4}|D|^2. \quad (10)$$

If the scatterer is at a typical location, namely, shifted from the center by $(\delta_x a_x, \delta_y a_y)$, with δ_x, δ_y , and δ_x/δ_y all irrational, the form factor is [12]

$$K(\tau) = 1 - \frac{|D|^2}{4}\tau + \frac{9}{128}|D|^4\tau^2 + \frac{81}{512}|D|^4\tau^3 - \frac{25}{1536}|D|^6\tau^3. \quad (11)$$

The difference between Eqs. (9) and (11) is due to length degeneracies. When the scatterer is at the center there are four diffracting segments of identical length, while if it is moved from the center this degeneracy is broken. For a quarter of all diffracting segments, for which N_j and M_j are odd, the degeneracy is totally lifted. For orbits with even N_j and M_j the location of the scatterer does not affect this degeneracy. For the rest of the segments the degeneracy is only partly lifted. When all length degeneracies are taken into account one obtains Eq. (11).

The form factor can be compared with numerical results obtained for the case of point interactions, where the eigenvalues are the roots of some function, and therefore can be easily found numerically [9]. The form factor was calculated for several values of D and compared to the analytical result (11) in Fig. 1. Agreement with Eq. (11) is found for short times, as is expected.

For a scatterer at the center only levels with wave functions that are symmetric with respect to the X and Y axes are perturbed by the scatterer. Since the value of these wave functions for all eigenvalues is the same $[\psi_n(\mathbf{x}=\mathbf{0}) = 2/\sqrt{A}]$, the resulting equation is the same as for the Soba billiard with periodic boundary conditions [11]. The form factor of the perturbed levels is related to that of the full

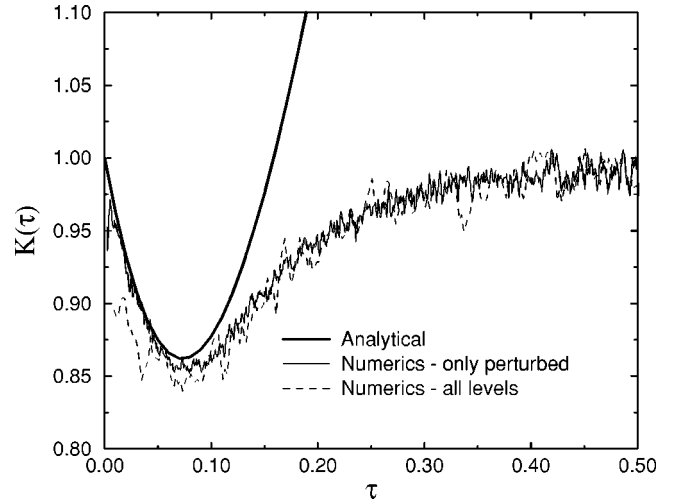


FIG. 2. The form factor for the scatterer at the center ($\mathbf{x}_0 = \mathbf{0}$) compared to the scaled form factor calculated from perturbed levels (12) and the analytical result (9).

spectrum. The eigenvalues of the four different symmetry classes of the rectangle can be assumed to be uncorrelated, leading to

$$K_{full}(\tau) = \frac{3}{4} + \frac{1}{4}K_{per}(4\tau). \quad (12)$$

The form factor calculated from all levels and the scaled form factor obtained from the perturbed levels with the help of Eq. (12) are compared to the analytical result (9), for $D = -4i$ in Fig. 2. It is clear that Eq. (12) is valid. For very small times the full form factor deviates from its expected value since there are not enough orbits that contribute (the calculation is not semiclassical enough).

In order to study the effect of the angle dependence of the diffraction on the form factor it was calculated to order τ^2 for the diffraction constant (2). For a scatterer at a typical location the form factor is found to be [12]

$$K(\tau) = 1 - \frac{1}{4}|C|^2\tau + \frac{9}{128}|C|^4\tau^2 + \dots \quad (13)$$

This form factor is similar to Eq. (11). Since C of (3) satisfies the optical theorem (10), this form factor can also be obtained from an angle independent potential, with the diffraction constant C . If the scatterer is at the center [12],

$$K(\tau) = 1 - \frac{1}{4}|C|^2\tau + \frac{1}{8}|C|^4C'\tau^2 + \dots \quad (14)$$

with $C' \equiv 1 - 2k^2 a^2 M_0$, where M_0 is related to integrals over the potential (1). It resembles the form factor (9) that was obtained for angle independent scattering. The modification is of the order $k^2 a^2$ and typically cannot change the sign of the expansion coefficients.

The condition for the applicability of the approximations used in this Brief Report is $a \ll \lambda \ll a_x, a_y$, where λ is the wavelength of the particles. Up to corrections of order $(ka)^3$,

the form factor (13) reduces to Eq. (11) in the order τ^2 . Therefore the angle dependence plays no role up to this order. If the scatterer is at the center the situation is somewhat different as can be seen by comparing Eq. (14) with Eq. (9). There is a correction C' resulting from the angular dependence of $D(\theta', \theta)$ given by (2). It is a consequence of the increased number of length degeneracies of the diffracting orbits when the scatterer is at the center. Since the form factor (13) describes essentially angle independent scattering the limit $a \rightarrow 0$ describes correctly the physics of the regime $a < \lambda$. This is so although the classical dynamics (in the long time limit) are expected to be chaotic in nearly all of phase space and similar to those of the Sinai billiard. This robustness improves the chances for the experimental realization of the results of the present work. For $a \gg \lambda$, semiclassical theory works and the system should behave as a Sinai billiard, with level statistics given by random matrix theory [3,5] (with deviations, see [25]).

The spectral statistics found in the present work differ from those of the known universality classes. They are characterized by a form factor of the type presented in Figs. 1

and 2. This form factor is equal to 1 at $\tau=0$, resulting from the fact that for small τ the number of classical orbits that are scattered is small. The contribution that is first order in τ originates from combinations of forward diffracting orbits and periodic orbits. These always have the same lengths, leading to the contribution $\text{Im}D\tau$. By the optical theorem (10) it is always negative. For $\tau \gg 1$ the form factor approaches unity because of the discreteness of the spectrum [5]. This general description should hold for other integrable systems that are perturbed similarly.

It is our great pleasure to thank E. Bogomolny and M. Sieber for inspiring, stimulating, detailed, and informative discussions and for informing us about their results prior to publication. We would like to thank also M. Aizenman, E. Akkermans, and R. E. Prange for critical and informative discussions. This research was supported in part by the U.S.-Israel Binational Science Foundation (BSF), by the National Science Foundation under Grant No. PHY99-07949, and by the Minerva Center of Nonlinear Physics of Complex Systems.

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